§ 11.6 Gradient Vector

#5. $f(x,y,z) = xe^{2yz}$, $P(3,0,2)$, $\vec{u} = \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

(a) Find the gradient of f

(b) Evaluate the gradient at P

(c) Find the rate of change of f at P in the direction of $\vec{u}$

Sol (a): $\nabla f = \langle f_x, f_y, f_z \rangle = \underline{\langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz} \rangle}$ ※

(b): $\nabla f(3,0,2) = \underline{\langle 1, 12, 0 \rangle}$ ※

(c) $D_{\vec{u}} f(3,0,2) = D_{\hat{u}} f(3,0,2) = \nabla f(3,0,2) \cdot \hat{u}$

$\uparrow$ 因 $|\vec{u}| = 1$ by 公式

$= \langle 1, 12, 0 \rangle \cdot \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle$

$= \underline{-\frac{22}{3}}$ ※

#7 $f(x,y) = 1 + 2x\sqrt{y}$, $P(3,4)$, $\vec{v} = \langle 4, -3 \rangle$

Find $D_{\vec{v}} f(3,4) = ?$

Sol: $\nabla f = \langle 2\sqrt{y}, 2x \cdot \frac{1}{2}\frac{1}{\sqrt{y}} \rangle = \langle 2\sqrt{y}, \frac{x}{\sqrt{y}} \rangle$

$\nabla f(3,4) = \langle 4, \frac{3}{2} \rangle$

$|\vec{v}| = 5$, $\hat{v} = \langle \frac{4}{5}, \frac{-3}{5} \rangle$

Thus $D_{\vec{v}} f(3,4) = \nabla f(3,4) \cdot \hat{v}$

$= \langle 4, \frac{3}{2} \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle = \underline{\frac{23}{10}}$ ※

#11. $g(x,y,z) = (x+2y+3z)^{\frac{3}{2}}$, $P(1,1,2)$, $\vec{v} = 2\hat{j} - \hat{k}$

Find $D_{\vec{v}} g(1,1,2) = ?$

Sol: $\nabla g = \langle \frac{3}{2}\sqrt{x+2y+3z}, 3\sqrt{x+2y+3z}, \frac{9}{2}\sqrt{x+2y+3z} \rangle$

$\nabla g(1,1,2) = \langle \frac{9}{2}, 9, \frac{27}{2} \rangle$, $|\vec{v}| = \sqrt{5}$, $\hat{v} = \langle 0, \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle$

$D_{\vec{v}} g(1,1,2) = \nabla g(1,1,2) \cdot \hat{v} = \langle \frac{9}{2}, 9, \frac{27}{2} \rangle \cdot \langle 0, \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle$

$= \underline{\frac{9}{2\sqrt{5}}}$ ※

#15. Find the maximum rate of change of $f(x,y) = \frac{y^2}{x}$ at the point $(2,4)$ and the direction in which it occurs.

sol: max rate of change $= |\vec{\nabla} f|$

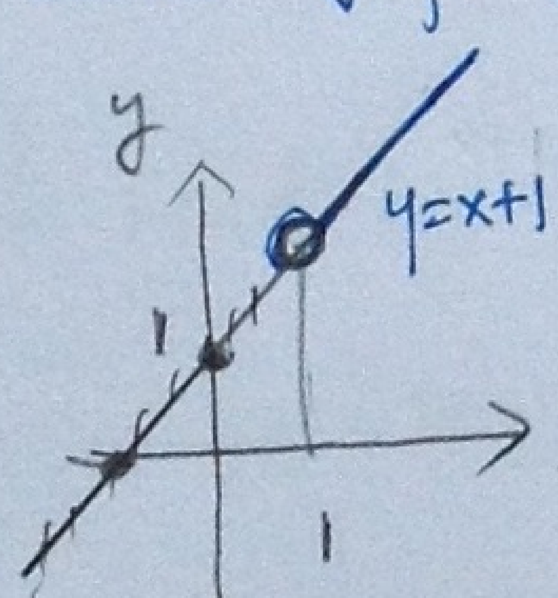
and the direction is $\vec{\nabla} f$

$$\vec{\nabla} f = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x} \right\rangle \Rightarrow \vec{\nabla} f(2,4) = \langle -4, 4\rangle = 4\langle -1, 1\rangle$$

Thus max rate of change $= \underline{4\sqrt{2}}$ ※ and in the direction $\underline{\langle -1, 1\rangle}$ ※

#21. Find all pts at which the direction of fastest change of the fct $f(x,y) = x^2+y^2-2x-4y$ is $\hat{i}+\hat{j}$

sol: $\vec{\nabla} f = \langle 2x-2, 2y-4\rangle = 2k\langle 1, 1\rangle$, $k>0$



$\begin{cases} x-1 = k>0 \\ y-2 = k \end{cases} \Rightarrow x-1 = y-2 \Rightarrow y = x+1$

消去k得

有損失一些條件,回去找 $x>1$ and $y>2$

Ans: $\underline{\{(x,y) \mid y = x+1 \text{ and } x>1\}}$ 如圖所示 ※

#25. $V(x,y,z) = 5x^2-3xy+xyz$ electrical potential

(a) Find the rate of change of the potential at $P(3,4,5)$ in $\vec{v} = \langle 1, 1, -1\rangle$ direction

(b) In which direction does V change most rapidly at P?

(c) What is the max rate of change at P?

sol: $\vec{\nabla} V = \langle 10x-3y+yz, xz-3x, xy\rangle$

$\vec{\nabla} V(3,4,5) = \langle 38, 6, 12\rangle$, $|\vec{v}| = \sqrt{3}$

(a) $D_{\vec{v}} V(3,4,5) = \vec{\nabla} V(3,4,5)\cdot\hat{v} = \langle 38,6,12\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right\rangle = \underline{\frac{32}{\sqrt{3}}}$ ※

(b) in direction of $\underline{\langle 19, 3, 6\rangle}$ ※

(c) $|\nabla \vec{V}(3,4,5)| = 2\sqrt{19^2+3^2+6^2} = \underline{2\sqrt{406}}$ ※

§11.6. Gradient Vector

#27. $f \in C^1$, $A(1,3)$, $B(3,3)$, $C(1,7)$ and $D(6,15)$

$D_{\overrightarrow{AB}} f(A) = 3$, $D_{\overrightarrow{AC}} f(A) = 26$.

Find $D_{\overrightarrow{AD}} f(A) = ?$

sol: $\overrightarrow{AB} = \langle 2, 0 \rangle$ unit vector is $\hat{i} = \langle 1, 0 \rangle$

$\Rightarrow D_{\overrightarrow{AB}} f(A) = D_{\hat{i}} f(A) = f_x(1,3) = 3$

$\overrightarrow{AC} = \langle 0, 4 \rangle$ unit vector is $\hat{j} = \langle 0, 1 \rangle$

$\Rightarrow D_{\overrightarrow{AC}} f(A) = f_y(1,3) = 26$

$\overrightarrow{AD} = \langle 5, 12 \rangle$ $|\overrightarrow{AD}| = 13$ unit vector $\hat{v} = \langle \frac{5}{13}, \frac{12}{13} \rangle$

$\therefore D_{\overrightarrow{AD}} f(1,3) = D_{\hat{v}} f(1,3) = \vec{\nabla} f(1,3) \cdot \hat{v} = \langle f_x(1,3), f_y(1,3) \rangle \cdot \hat{v}$

$= \langle 3, 26 \rangle \cdot \langle \frac{5}{13}, \frac{12}{13} \rangle = \underline{25\frac{2}{13}}$ or $\frac{327}{13}$ ※

#31 Find eqs of (a) the tangent plane and (b) the normal line to the surface $x^2 - 2y^2 + z^2 + yz = 2$ at $(2,1,-1)$ pt.

sol: Let $f(x,y,z) = x^2 - 2y^2 + z^2 + yz$

Thus the surface is the level surface of $w = f(x,y,z)$ (with level $= 2$)

$\vec{\nabla} f = \langle 2x, -4y + z, 2z + y \rangle$

$\vec{\nabla} f(2,1,-1) = \langle 4, -5, -1 \rangle$ is a normal of the (level) surface

(a) eq of E: $\underline{4x - 5y - z = 4}$ ※

(b) eq of L: $\dfrac{x-2}{4} = \dfrac{y-1}{-5} = \dfrac{z+1}{-1}$

or $\underline{x = 2 + 4t,\ y = 1 - 5t,\ z = -1 - t}$ ※

§11.6 Gradient Vector

#37 $f(x,y) = x^2+4y^2$, find $\vec{\nabla} f(2,1)$ and tangent line to the level curve $f(x,y)=8$ at $(2,1)$. Sketch the level curve, the tangent line, and gradient vector.

sol: $\vec{\nabla} f(x,y) = \langle 2x, 8y \rangle$, $\underline{\vec{\nabla} f(2,1) = \langle 4, 8 \rangle}$ #

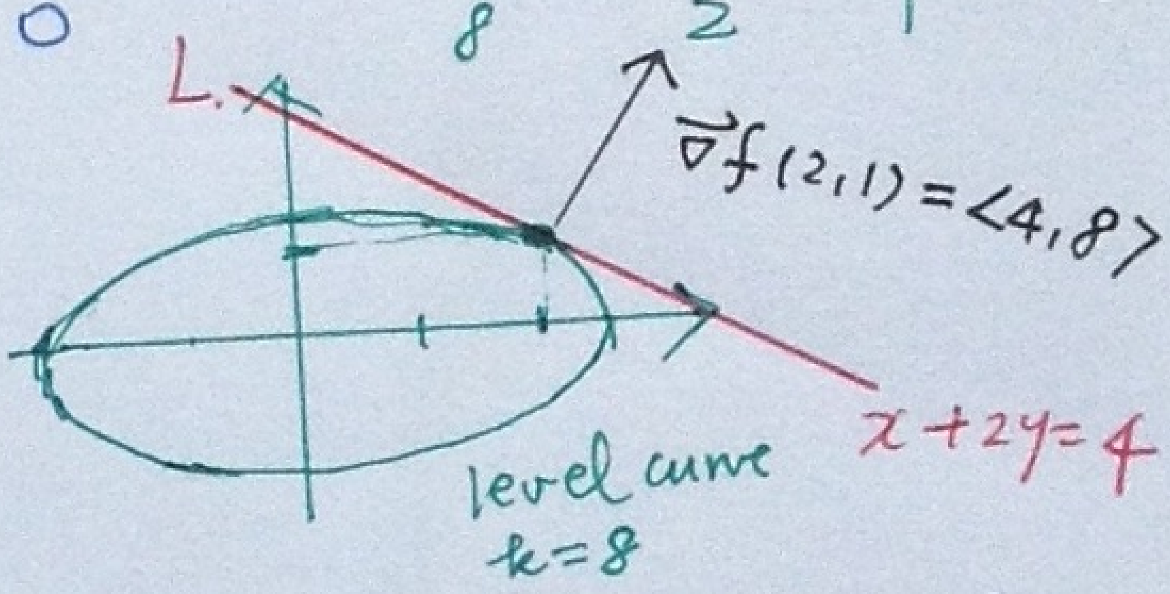
$\because f(2,1)=8 \quad \therefore \vec{\nabla} f(2,1)$ is a normal vector of the level curve $\{(x,y) \mid f(x,y)=8\}$ at $(2,1)$.

$x^2+4y^2=8$

$\frac{x^2}{8} + \frac{y^2}{2} = 1$

eq of L: $\langle 1, 2 \rangle \cdot \langle x-2, y-1 \rangle = 0$

$\Rightarrow \underline{x+2y=4}$ #



#41. Find the pt on $S: x^2-y^2+2z^2=1$ where normal line $\parallel \overleftrightarrow{AB}$, $A(3,-1,0)$, $B(5,3,6)$

sol: Let $f(x,y,z) = x^2-y^2+2z^2$

Then above surface S is a level surface of f

Thus $\vec{\nabla} f$ is a normal vector of S

Since $\overrightarrow{AB} = \langle 2, 4, 6 \rangle$ and $\vec{\nabla} f(x,y,z) = \langle 2x, -2y, 4z \rangle$

Then $\langle x, -y, 2z \rangle \parallel \langle 1, 2, 3 \rangle$

$$\begin{cases} \langle x, -y, 2z \rangle = k \langle 1, 2, 3 \rangle, & k \in \mathbb{R},\ k \neq 0 \\ x^2-y^2+2z^2 = 1 \end{cases}$$

$\Rightarrow k = \pm\frac{\sqrt{6}}{3}$ and there are two such pt $\underline{\left(\pm\frac{\sqrt{6}}{3}, \mp\frac{2\sqrt{6}}{3}, \pm\frac{\sqrt{6}}{2}\right)}$ #

#45. Find parametric eq for tangent line L to the curve C at $(-1,1,2)$

C: S_1: $z = x^2+y^2$ intersects S_2: $4x^2+y^2+z^2=9$

sol: L lies on tangent plane of S_1 and S_2.

S_1 之 normal is $\vec{n_1} = \langle 2x, 2y, -1 \rangle \big|_{(-1,1,2)} = \langle -2, 2, -1 \rangle$

S_2 之 normal is $\vec{n_2} = \langle 8x, 2y, 2z \rangle \big|_{(-1,1,2)} = \langle -8, 2, 4 \rangle$, 取成 $\langle -4, 1, 2 \rangle$

$\therefore$ directional vector of $L = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & -1 \\ -4 & 1 & 2 \end{vmatrix} = \langle 5, 8, 6 \rangle$

eq of L: $\underline{x=-1+5t,\ y=1+8t,\ z=2+6t}$ #